Water vapor transport to a semi-infinite material with simultaneous varying surface relative humidity and temperature

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Water vapor transport to a semi-infinite material with simultaneous varying surface relative humidity and temperature

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Abstract. The water vapor transfer between the indoor air and hygroscopic finishing materials is of importance for the moisture balance of the room. Most protocols for determining the effect are based on isothermal conditions and cycling relative humidity in the form of square wave or sinusoidal functions. A new analytical solution for a material exposed to a both time varying surface relative humidity and temperature is presented in the paper. The time varying temperature inside the material is assumed to follow the surface temperature throughout the material layer since the reaction time for temperature changes in a reasonable thin surface material is rather short compared with the one for moisture changes. The semi-infinite approach is justified by the fact that the penetration depth for moisture variations are very limited for diurnal variations. The analytical approach and solution are presented in the paper

1 Introduction

Moisture uptake in the interior surface materials due to varying humidity in the indoor air is of importance both for the humidity levels of the room itself and for the moisture conditions in the surface materials. There have been research on this topic [1-3] and standards [4] and some ongoing measurements [5].

However, there has been less focus on cases with both a varying RH and temperature. Some recent results on this are presented in [6-7]. This study gives new handy analytical expressions for the effect of combined variations in RH and temperature. In parallel to this paper, an experimental validation of the model is under way.

2 Governing equations

In this study the surface vapor resistance is neglected which means that the surface RH is always equal to the room RH. This will represent the case of maximum interaction.

2.1 Equations

The moisture balance equation reads:

$$-\frac{\partial}{\partial x}\left(-\delta_v \frac{\partial v}{\partial x}\right)=\frac{\partial w}{\partial t} \tag{1}$$

Here, v (kg/m^3) is the humidity by volume and w (kg/m^3) is the moisture content.

The analysis in this paper assumes that the temperature of the surface material always follows the interior temperature without any delay. This is a reasonable assumption since temperature changes are much more rapid than moisture changes and that it is only the thin interior surface layer that is affected by variations in indoor cyclic moisture variations.

2.2 Simplified equations

Two simplifications will be introduced. The first one is that the vapor diffusion coefficient δ_v (m^2/s) is constant:

$$\delta_v = \delta_v^0 \tag{2}$$

The second simplification is that the slope of the sorption curve is constant. Furthermore, hysteresis is neglected.

$$\frac{\partial w}{\partial \varphi}=\xi \tag{3}$$

The moisture balance equation then becomes:

$$\delta_v^0 \frac{\partial^2 v}{\partial x^2}=\delta_v^0 v_s(T)\frac{\partial^2 \varphi}{\partial x^2}=\frac{\partial w}{\partial t}=\frac{\partial w}{\partial \varphi}\frac{\partial \varphi}{\partial t}=\xi\frac{\partial \varphi}{\partial t} \tag{4}$$

Here, v_s is the humidity by volume at saturation.
Introducing the vapor moisture diffusivity a_v (m^2/s):

$$\frac{\partial^2 \varphi}{\partial x^2}=\frac{1}{a_v}\frac{\partial \varphi}{\partial t} \qquad a_v=\frac{\delta_v^0 v_s(T)}{\xi} \tag{5}$$

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3 Two basic solutions for constant temperature and zero surface resistance

3.1 Step change

A material surface layer with initial relative humidity of φ_0 and temperature of T_0 is considered.

We have a RH step change at the boundary, *x=0,* at time zero. The surface RH is:

$$\varphi_s = \varphi_0 + \Delta\varphi \tag{6}$$

The analytical solution [1] for a semi-infinite domain $x \geq 0, t \geq 0$ is:

$$\varphi(x,t) = \varphi_0 + \Delta\varphi \cdot \text{erfc}\left(\frac{x}{\sqrt{4a_v t}}\right) \tag{7}$$

Here, erfc is the complimentary error function. The penetration depth, i.e. the depth were we approximately find half the disturbance of what happened at the boundary has propagated is:

$$x_{0.5} = \sqrt{a_v t} \tag{8}$$

Example: For wood this depth is around 0.0005 m after 1h, 0.0007 m after 2h, 0.003 m after 1 day and 0.007 m after a week.

3.2 Periodic change

If we have the following variation of the RH at the material surface, *x=0*:

$$\varphi_s = \varphi_0 + \varphi_A \cdot \cos\left(\frac{2\pi t}{t_p}\right) \tag{9}$$

Here, t_p (s) is the time period of the cosinusoidal variation. For the steady periodic case, the following well-known expression [1] for a semi-infinite domain $x \geq 0$ reads:

$$\varphi(x,t) = \varphi_0 + \varphi_A \cdot e^{-x/d_{pv}} \cos\left(\frac{2\pi t}{t_p} - \frac{x}{d_{pv}}\right) \tag{10}$$

The periodic penetration depth d_{pv} (m), i.e. the depth were the amplitude of the RH has diminished with a factor $e^{-1} \approx 0.37$ is:

$$d_{pv} = \sqrt{\frac{a_v t_p}{\pi}} \tag{11}$$

Example: For wood this depth is around 0.001-0.002 m for a diurnal variation (t_p =24h) and 0.03 m for a yearly one.

These numbers show that the penetration depth of moisture in to the material is very small for diurnal variations.

This periodic variation is of particular interest in this paper. The moisture flow g (kg/m²/s) into the material at x=0 becomes:

$$g = g_A \cdot \cos\left(\frac{2\pi t}{t_p} + \frac{\pi}{4}\right)$$

$$g_A = \varphi_A \frac{\sqrt{2\pi a_v}}{\sqrt{t_p}} \xi \tag{12}$$

$$\left(g_A = \varphi_A d_{pv} \xi \sqrt{2} \frac{\pi}{t_p} \right)$$

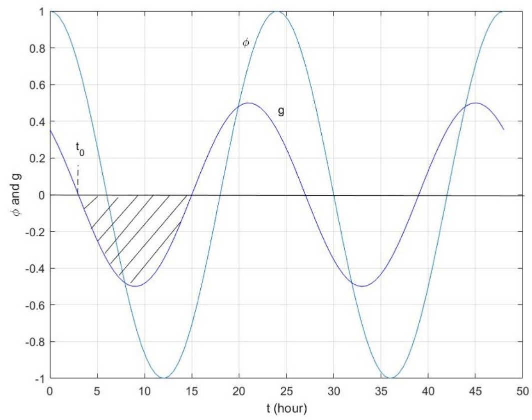

Fig 1. Principle sketch of the diurnal boundary RH variation (9) and the moisture flow into the material (12). The total moisture uptake during a half period, m_A, is indicated by the hatched area. The time t_0 is also marked.

We are interested in the total uptake of moisture by the surface during a half period. The magnitude of the moisture uptake m_A (kg/m²) from time t_0 to time $t_0+t_p/2$ is:

$$m_A = \left| \int_{t_0}^{t_0+t_p/2} g(t')dt' \right| \tag{13}$$

The time t_0 is chosen so that the net uptake of moisture is zero at this time, see Fig 1. For the periodic RH variation according to (9) and (12) it is equal to $t_p/8$.

For steady periodic variations the net accumulation of moisture in the material is on average zero over time. The amplitude of the integrated mass flow (kg/m²), (13), into and out from the material surface during a half period, [1]:

$$m_A = g_A \cdot \frac{t_p}{\pi} = \varphi_A d_{pv} \xi \sqrt{2}$$
$$m_A = \frac{\varphi_A}{2\sqrt{\pi}} \xi \sqrt{a_v t_p} \cdot f_A \quad f_A = 2\sqrt{2} \tag{14}$$

4 Varying temperature

We will consider the case of time varying temperature:

$$\frac{\partial^2\varphi}{\partial x^2}=\frac{1}{a_v(t)}\frac{\partial\varphi}{\partial t}\qquad a_v(t)=\frac{\delta_v^0 v_s(T(t))}{\xi} \tag{15}$$

Here, the material temperature is equal to the boundary temperature. It is purely a function of time:

$$\frac{\partial^2\varphi}{\partial x^2}=\frac{1}{a_v(t)}\frac{\partial\varphi}{\partial t} \tag{16}$$

A variable substitution is introduced:

$$\tau(t)=\int_0^t a_v(t')\,dt' \tag{17}$$

The moisture balance equation is transformed to:

$$\frac{\partial^2\varphi}{\partial x^2}=\frac{\partial\varphi}{\partial\tau} \tag{18}$$

Here, we have an equation that is similar to the one-dimensional moisture balance equation with the diffusivity a_v equal to 1. The equation is linear when using this transformed time variable and superposition techniques can be used. We therefore only need the basic solution for a unit-step change or sinusoidal variations to handle complex time-wise changes in the RH. For a unit step change we for instance have:

$$\begin{cases}\varphi(x,\tau)=\varphi_0+\Delta\varphi\cdot\operatorname{erfc}\left(\dfrac{x}{\sqrt{4\tau}}\right)\\ x\ge 0,\tau\ge 0\end{cases} \tag{19}$$

And for a cosinusoidal variation (steady-periodic) in the τ regime with the time period τ_p:

$$\varphi(x,\tau)=\varphi_0+\varphi_A\cdot e^{-x/d_{pv}}\cos\left(\frac{2\pi\tau}{\tau_p}-\frac{x}{d_{pv}}\right)\qquad x\ge 0\qquad d_{pv}=\sqrt{\frac{\tau_p}{\pi}} \tag{20}$$

5 Wave train

5.1 RH in the material layer

Assume a periodic square shaped boundary RH and temperature in time (corresponding to a wave train symmetrical around $t=0$) with the time period t_p. For simplicity we choose that half of the time RH has its highest value, $\varphi_0+\varphi_A$, and temperature T_1 and the other half of the time the lowest one, $\varphi_0-\varphi_A$, and temperature T_2. Alternative proportions can be studied using the same procedure as below.

$$\varphi_s=\begin{cases}\varphi_0+\varphi_A & 0\le t\le\dfrac{t_p}{4}\quad \dfrac{3t_p}{4}\le t\le t_p\\ \varphi_0-\varphi_A & \dfrac{t_p}{4}\le t\le\dfrac{3t_p}{4}\end{cases}\qquad T_s=\begin{cases}T_1\quad (a_v=a_{v1}) & 0\le t\le\dfrac{t_p}{4}\quad \dfrac{3t_p}{4}\le t\le t_p\\ T_2\quad (a_v=a_{v2}) & \dfrac{t_p}{4}\le t\le\dfrac{3t_p}{4}\end{cases} \tag{21}$$

When transformed to the τ-variable we get the high RH value $\varphi_0+\varphi_A$ and temperature T_1 for the following intervals:

$$0\le\tau\le\tau\left(\frac{t_p}{4}\right)\quad \tau\left(\frac{3t_p}{4}\right)\le\tau\le\tau(t_p) \tag{22}$$

, and low RH value $\varphi_0-\varphi_A$ and temperature T_2 for:

$$\tau\left(\frac{t_p}{4}\right)\le\tau\le\tau\left(\frac{3t_p}{4}\right) \tag{23}$$

From the definition of τ (17):

$$\begin{gathered}\tau\left(\frac{t_p}{4}\right)=\frac{a_{v1}\cdot t_p}{4}\\ \tau\left(\frac{3t_p}{4}\right)=\frac{a_{v1}\cdot t_p}{4}+\frac{a_{v2}\cdot t_p}{2}\\ \tau_p=\tau(t_p)=\frac{a_{v1}+a_{v2}}{2}t_p=\bar{a}_v\cdot t_p\end{gathered} \tag{24}$$

Figure 1 shows the boundary values as functions of t and τ.

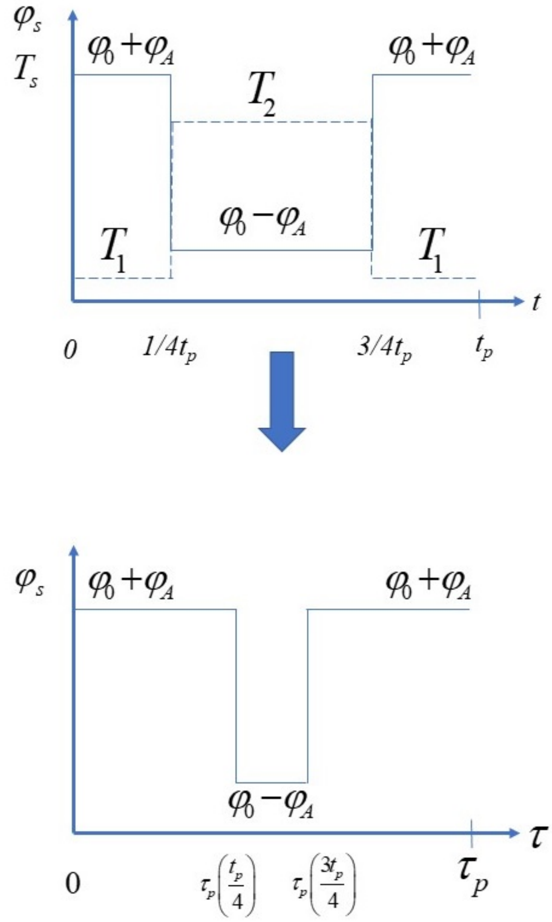

Fig 2. Boundary RH and temperature as functions of t and τ .

The fraction, α, of the time period τ_p when the RH has its highest value (and $a_v = a_{v1}$) is:

$$\alpha = \frac{a_{v1}}{a_{v1}+a_{v2}} \qquad \left(= \frac{a_{v1}\cdot t_p/2}{\left(a_{v1}+a_{v2}\right)\cdot t_p/2} \right) \tag{25}$$

Using Fourier series in τ regime, the wave train can be written as:

$$\varphi_s = \varphi_0 + (2\alpha-1)\cdot\varphi_A + \\ +\frac{4\varphi_A}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi\alpha)}{n}\cos\left(2n\pi\frac{\tau(t)}{\tau_p}\right) \tag{26}$$

When α=0.5, the high and low values of a_v are equal and the second term will vanish. This represent the isothermal ($a_{v1} = a_{v2}$). For other values of α there will be a change in the average value of RH in the material from φ_0.

Since we know the solution for the heat conduction equation for cosinusoidal boundary conditions (20) we get:

$$\varphi(x,t) = \varphi_0 + (2\alpha-1)\cdot\varphi_A + \\ +\frac{4\varphi_A}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi\alpha)}{n}e^{-x/d_{pv,n}}\cos\left(2n\pi\frac{\tau(t)}{\tau_p}-\frac{x}{d_{pv,n}}\right) \tag{27}$$

$$d_{pv,n} = \sqrt{\frac{\tau_p}{n\pi}}$$

For the isothermal case ($a_{v1} = a_{v2}$)

$$\varphi(x,t) = \varphi_0 + \\ +\frac{4\varphi_A}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n}e^{-x/d_{pv,n}}\cos\left(2n\pi\frac{t}{t_p}-\frac{x}{d_{pv,n}}\right)$$

$$d_{pv,n} = \sqrt{\frac{a_v t_p}{n\pi}} \tag{28}$$

Figure 3 shows the relative amplitude of RH and the average RH as a function of depth from the surface using (27) for a specific case.

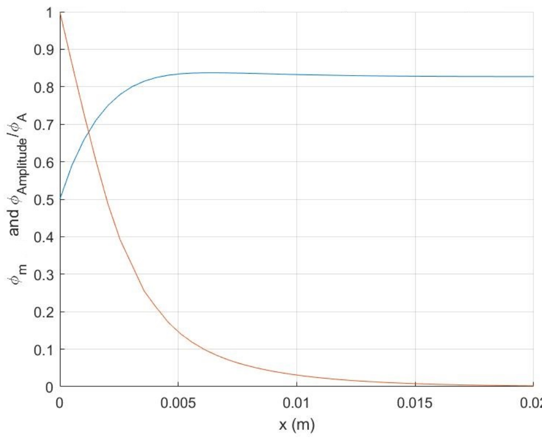

Fig 3. Example of the relative amplitude of RH and the average RH as a function of depth from the surface.
$\varphi_0 = 0.5$ $\varphi_A = 0.4$ $a_{v1} = 10\cdot10^{-10}\,\mathrm{m^2/s}$ $a_{v2} = 1\cdot10^{-10}\,\mathrm{m^2/s}$

5.2 Moisture uptake

The moisture flow g (kg/m^2/s) into the material is:

$$g(t) = -\delta_v^0 v_s(T(t))\left.\frac{\partial\varphi(x,t)}{\partial x}\right|_{x=0} = \\ = \varphi_A\delta_v^0 v_s(T(t))\sqrt{2}\,\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi\alpha)}{n\cdot d_{pv,n}}\cos\left(2n\pi\frac{\tau(t)}{\tau_p}+\frac{\pi}{4}\right) = \\ = \varphi_A\sqrt{2\pi}\,\xi\frac{a_v(t)}{\sqrt{\tau_p}}\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi\alpha)}{\sqrt{n}}\cos\left(2n\pi\frac{\tau(t)}{\tau_p}+\frac{\pi}{4}\right) = \tag{29}$$

For the isothermal case ($\alpha = 1/2$):

$$g(t) = \varphi_A\sqrt{2\pi}\,\xi\sqrt{a_v}\cdot\frac{1}{\sqrt{t_p}} \\ \cdot\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{\sqrt{2m+1}}\cos\left(2\cdot(2m+1)\pi\frac{t}{t_p}+\frac{\pi}{4}\right) \tag{30}$$

The integrated m (kg/m^2) moisture uptake from time zero to time t is:

$$m(t)=\int_0^t g(t')dt' \tag{31}$$

The formula can be reformulated using a variable substitution, $s=\tau(t)$:

$$m(\tau(t))=\frac{\varphi_A}{2\sqrt{\pi}}\xi\sqrt{\tau_p}\cdot f_m$$
$$f_m= \tag{32}$$
$$=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi\alpha)}{n\sqrt{n}}\left[\sqrt{2}\cdot\sin\left(2n\pi\frac{\tau(t)}{\tau_p}+\frac{\pi}{4}\right)-1\right]$$

Figure 4 shows f_m, (32), for a case. As seen from the figure the moisture is leaving the material at time t_0 until time $t_0+t_p/2$ when the moisture uptake is increasing again. The total moisture uptake/release during a half cycle is equal m_A, which is illustrated in Fig 4.

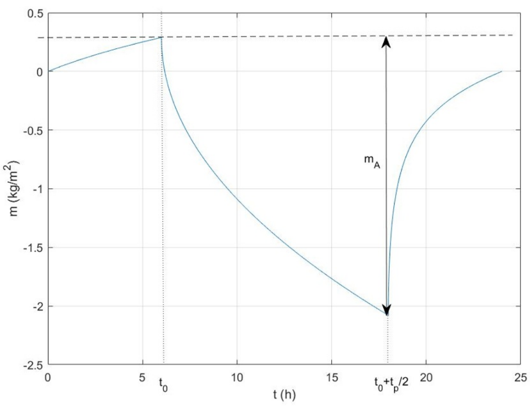

Fig 4. Parameter f_m, (32), for the integrated moisture uptake as a function of time t during a day. Diurnal variations (t_p=24h).

$a_{v1}=10\cdot10^{-10}\,\mathrm{m^2/s}$ $a_{v2}=1\cdot10^{-10}\,\mathrm{m^2/s}$

The moisture uptake during a half period determined by (13) and (31-32) becomes:

$$m_A=\frac{\varphi_A}{2\sqrt{\pi}}\xi\sqrt{\tau_p}\cdot f_A \tag{33}$$

The time lag t_0 and the amplitude function f_A is practically found by integrating the moisture flow over time $m(t)$, (31). The amplitude m_A is then obtained from the difference between m_{max} and m_{min}. See Fig 4. For the studied case the time t_0 is approximately equal to $t_p/4$, i.e the time when RH goes from a high to a low value.

The amplitude parameter f_A of the integrated mass flow into and out from the material surface (33) is given in Table 1.

Table 1. The amplitude parameter f_A for the determination of the total moisture uptake during a half cycle.

α	$f_A(\alpha)$
0	0
0.0001	0.0802
0.001	0.2541
0.01	0.8034
0.05	1.777
0.10	2.479
0.15	2.982
0.20	3.369
0.25	3.672
0.30	3.907
0.35	4.083
0.40	4.205
0.45	4.277
0.50	4.300

Due to symmetry we will get the same amplitude of the integrated mass flow in to the material if let a_{v1} occur at the high boundary RH and a_{v2} at the low one:

$$f_A(\alpha)=f_A(1-\alpha) \tag{34}$$

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5.3 Example

The material surface is made of spruce:

$$\delta_v^0 = 1.5 \cdot 10^{-6}\ \mathrm{m^2/s} \qquad \xi = 100\ \mathrm{kg/m^3}$$

The RH will be high when the temperature is low and low when the temperature is high. The following boundary values are assumed.

$$\varphi_0 = 0.4 \quad \varphi_A = 0.3 \quad t_p = 24 \cdot 3600\ \mathrm{s}$$
$$T_1 = 18\,°\mathrm{C} \quad T_2 = 25\,°\mathrm{C}$$

We get the following vapor diffusivities:

$$a_{v1} = \frac{\delta_v^0 v_s(T_1)}{\xi} = \frac{1.5 \cdot 10^{-6} \cdot v_s(18)}{100} = 2.304 \cdot 10^{-10}$$
$$a_{v2} = \frac{\delta_v^0 v_s(T_2)}{\xi} = \frac{1.5 \cdot 10^{-6} \cdot v_s(25)}{100} = 3.455 \cdot 10^{-10}$$

The α parameter (25) becomes:

$$\alpha = \frac{a_{v1}}{a_{v1} + a_{v2}} = \frac{2.304}{2.304 + 3.455} = 0.4$$

$$\overline{\varphi} = \varphi_0 + (2\alpha - 1) \cdot \varphi_A = 0.34$$

The average RH in the material will be lower than the average of the boundary due to changes of the temperature over time.

The magnitude of the integrated mass flow into and out from the material surface is:

$$m_A = \frac{0.3}{2\sqrt{\pi}} 100 \sqrt{\frac{2.304 + 3.455}{2} 10^{-10} \cdot 24 \cdot 3600} \cdot f_A(0.4)$$
$$m_A = 0.0422 \cdot 4.205 = 0.178 \quad \mathrm{kg/m^2}$$

6 Conclusions

Explicit analytical formulas for the moisture distribution inside a material that is exposed to cyclic variations in RH and temperature are derived. The boundary RH and temperature have different values during each half period. The formula can be used to estimate the uptake and release of moisture to an exposed material surface as long as the periodic penetration depth is much smaller than the thickness of the material.

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